

t -singular linear spaces

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Abstract

As a generalization of singular linear spaces, we introduce the concept of t -singular linear spaces, make some Anzahl formulas of subspaces, and determine the suborbits of t -singular linear groups.

Keywords: t -singular linear space, suborbit

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1. Introduction

Let \mathbb{F}_q be a finite field with q elements, where q is a prime power. For a non-negative integer n , \mathbb{F}_q^n denotes the n -dimensional row vector space over \mathbb{F}_q . The set of all $(n_1 + n_2) \times (n_1 + n_2)$ nonsingular matrices over \mathbb{F}_q

$$\begin{pmatrix} & n_1 & & n_2 \\ T_{11} & T_{12} & & \\ 0 & T_{22} & & \end{pmatrix}_{n_1 \atop n_2}$$

forms a group under matrix multiplication, called the *singular general linear group* of degree $n_1 + n_2$ over \mathbb{F}_q and denoted by $GL_{n_1, n_2}(\mathbb{F}_q)$. The vector space $\mathbb{F}_q^{n_1 + n_2}$ together with the right multiplication action of $GL_{n_1, n_2}(\mathbb{F}_q)$ is called the $(n_1 + n_2)$ -dimensional *singular linear space* over \mathbb{F}_q .

As a generalization of attenuated spaces, the concept of singular linear spaces was firstly introduced in [8]. In [9, 10], we obtained some Anzahl formulas of subspaces in singular linear spaces and discussed their applications to association schemes. In this paper, we generalize the concept of singular linear spaces to that of t -singular linear spaces. In Section 2, we introduce the concept of t -singular linear spaces and give some Anzahl formulas of subspaces. In Section 3, we determine the suborbits of t -singular linear groups.

2. t -singular linear spaces

This paper involves partitioned matrices whose entries are themselves submatrices. For typographical convenience, we often leave blank the zero submatrices. We write $I^{(r)}$ for the identity matrix of size r , and we omit r if it is clear from the context.

For non-negative integers n_1, n_2, \dots, n_t , the set of all nonsingular matrices over \mathbb{F}_q

$$\begin{pmatrix} & n_1 & & n_2 & & & n_t \\ T_{11} & T_{12} & \cdots & & & & T_{1t} \\ & T_{22} & \cdots & & & & T_{2t} \\ & & & \ddots & & & \vdots \\ & & & & T_{tt} & & \end{pmatrix}_{n_1 \atop n_2 \atop n_t}$$

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forms a group under matrix multiplication, called the t -singular general linear group of degree $n_1 + n_2 + \cdots + n_t$ over \mathbb{F}_q and denoted by $GL_{n_1, n_2, \dots, n_t}(\mathbb{F}_q)$.

Let P be an m -dimensional subspace of $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$, denote also by P an $m \times (n_1 + n_2 + \cdots + n_t)$ matrix of rank m whose rows span the subspace P and call the matrix P a matrix representation of the subspace P . There is an action of $GL_{n_1, n_2, \dots, n_t}(\mathbb{F}_q)$ on $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$ defined as follows:

$$\begin{aligned} \mathbb{F}_q^{n_1+n_2+\cdots+n_t} \times GL_{n_1, n_2, \dots, n_t}(\mathbb{F}_q) &\longrightarrow \mathbb{F}_q^{n_1+n_2+\cdots+n_t} \\ ((x_1, \dots, x_{n_1+n_2+\cdots+n_t}), T) &\longmapsto (x_1, \dots, x_{n_1+n_2+\cdots+n_t})T. \end{aligned}$$

The above action induces an action on the set of subspaces of $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$; i.e., a subspace P is carried by $T \in GL_{n_1, n_2, \dots, n_t}(\mathbb{F}_q)$ to the subspace PT . The vector space $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$ together with the above group action is called the $(n_1 + n_2 + \cdots + n_t)$ -dimensional t -singular linear space over \mathbb{F}_q . Note that 2-singular linear spaces are just the singular linear spaces.

For $1 \leq j \leq n_1 + n_2 + \cdots + n_t$, let e_j be the row vector in $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$ whose j th coordinate is 1 and all other coordinates are 0. For $2 \leq i \leq t$, denote by E_i the $(n_i + n_{i+1} + \cdots + n_t)$ -dimensional subspace of $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$ generated by $e_{n_1+\cdots+n_{i-1}+1}, e_{n_1+\cdots+n_{i-1}+2}, \dots, e_{n_1+n_2+\cdots+n_t}$. A k_1 -dimensional subspace P of $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$ is called a subspace of type (k_1, k_2, \dots, k_t) if $\dim(P \cap E_i) = k_i$ for each $2 \leq i \leq t$.

Let $\mathcal{M}(k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t)$ denote the set of all the subspaces of type (k_1, k_2, \dots, k_t) of $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$ and let $N(k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t)$ be the size of $\mathcal{M}(k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t)$.

Theorem 2.1. *The set $\mathcal{M}(k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t)$ is non-empty if and only if*

$$0 \leq k_i - k_{i+1} \leq n_i \quad (1 \leq i \leq t-1) \text{ and } 0 \leq k_t \leq n_t. \quad (1)$$

Moreover, if (1) holds, then $\mathcal{M}(k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t)$ forms an orbit under $GL_{n_1, n_2, \dots, n_t}(\mathbb{F}_q)$ and

$$N(k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t) = \begin{bmatrix} n_t \\ k_t \end{bmatrix}_q \prod_{j=1}^{t-1} q^{(k_j - k_{j+1})(n_{j+1} + \cdots + n_t - k_{j+1})} \begin{bmatrix} n_j \\ k_j - k_{j+1} \end{bmatrix}_q.$$

PROOF. The first statement is trivial.

Now suppose (1) holds. Pick any subspace P of type (k_1, k_2, \dots, k_t) in $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$. Then P has a matrix representation

$$\begin{pmatrix} \begin{matrix} n_1 & n_2 & & n_{t-1} & n_t \\ P_{11} & P_{12} & \cdots & P_{1,t-1} & P_{1t} \\ & P_{22} & \cdots & P_{2,t-1} & P_{2t} \\ & & \ddots & \vdots & \vdots \\ & & & P_{t-1,t-1} & P_{t-1,t} \\ & & & & P_{tt} \end{matrix} & \begin{matrix} k_1 - k_2 \\ k_2 - k_3 \\ \vdots \\ k_{t-1} - k_t \\ k_t \end{matrix} \end{pmatrix}, \quad (2)$$

where $\text{rank } P_{ii} = k_i - k_{i+1}$ ($1 \leq i \leq t-1$) and $\text{rank } P_{tt} = k_t$. By basic facts in linear algebra, there exists a $T \in GL_{n_1, n_2, \dots, n_t}(\mathbb{F}_q)$ such that PT has the matrix representation

$$\begin{pmatrix} \begin{matrix} n_1 & n_2 & & n_{t-1} & n_t \\ (I \ 0) & & & & \\ & (I \ 0) & & & \\ & & \ddots & & \\ & & & (I \ 0) & \\ & & & & (I \ 0) \end{matrix} & \begin{matrix} k_1 - k_2 \\ k_2 - k_3 \\ \vdots \\ k_{t-1} - k_t \\ k_t \end{matrix} \end{pmatrix}. \quad (3)$$

Hence, $\mathcal{M}(k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t)$ forms an orbit under $GL_{n_1, n_2, \dots, n_t}(\mathbb{F}_q)$.

Denote by $n(k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t)$ the number of all matrices of the form (2). Suppose that P and Q represent the same subspace, then there is a $k_1 \times k_1$ nonsingular matrix U such that $P = UQ$. It follows that U is necessarily of the form

$$U = \begin{pmatrix} U_{11} & U_{12} & \cdots & U_{1t} \\ & U_{22} & \cdots & U_{2t} \\ & & \ddots & \vdots \\ & & & U_{tt} \end{pmatrix} \in GL_{k_1-k_2-k_3-\dots-k_{t-1}-k_t, k_t}(\mathbb{F}_q).$$

Moreover, if $UQ = Q$, then $U = I$. Consequently,

$$n(k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t) = |GL_{k_1-k_2-k_3-\dots-k_{t-1}-k_t, k_t}(\mathbb{F}_q)|N(k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t).$$

By [6, Lemma 1.5], the desired result follows. \square

For a fixed subspace P of type (k_1, k_2, \dots, k_t) in $\mathbb{F}_q^{n_1+n_2+\dots+n_t}$, let $\mathcal{M}(l_1, l_2, \dots, l_t; k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t)$ be the set of all the subspaces of type (l_1, l_2, \dots, l_t) contained in P , and let

$$N(l_1, l_2, \dots, l_t; k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t) = |\mathcal{M}(l_1, l_2, \dots, l_t; k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t)|.$$

By the transitivity of $GL_{n_1, n_2, \dots, n_t}(\mathbb{F}_q)$ on the set of subspaces of the same type, $N(l_1, l_2, \dots, l_t; k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t)$ is independent of the particular choice of the subspace P of type (k_1, k_2, \dots, k_t) .

Proposition 2.2. *The set $\mathcal{M}(l_1, l_2, \dots, l_t; k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t)$ is non-empty if and only if*

$$0 \leq l_i - l_{i+1} \leq k_i - k_{i+1} \leq n_i \quad (1 \leq i \leq t-1) \text{ and } 0 \leq l_t \leq k_t \leq n_t. \quad (4)$$

Moreover, if (4) holds, then

$$N(l_1, l_2, \dots, l_t; k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t) = \begin{bmatrix} k_t \\ l_t \end{bmatrix}_q \prod_{j=1}^{t-1} q^{(l_j - l_{j+1})(k_{j+1} - l_{j+1})} \begin{bmatrix} k_j - k_{j+1} \\ l_j - l_{j+1} \end{bmatrix}_q. \quad (5)$$

PROOF. The first statement is trivial.

Suppose (4) holds. By the transitivity of $GL_{n_1, n_2, \dots, n_t}(\mathbb{F}_q)$ on the set of subspaces of the same type, we may pick the subspace P of type (k_1, k_2, \dots, k_t) as the form (3). Since the number of subspaces of type (l_1, l_2, \dots, l_t) in $\mathbb{F}_q^{n_1+n_2+\dots+n_t}$ contained in P is equal to the number of subspaces of type (l_1, l_2, \dots, l_t) in $\mathbb{F}_q^{(k_1-k_2)+(k_2-k_3)+\dots+(k_{t-1}-k_t)+k_t}$, by Lemma 2.1, (5) holds. \square

For a fixed subspace P of type (l_1, l_2, \dots, l_t) in $\mathbb{F}_q^{n_1+n_2+\dots+n_t}$, let $\mathcal{M}'(l_1, l_2, \dots, l_t; k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t)$ be the set of all the subspaces of type (k_1, k_2, \dots, k_t) containing P , and let

$$N'(l_1, l_2, \dots, l_t; k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t) = |\mathcal{M}'(l_1, l_2, \dots, l_t; k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t)|.$$

By the transitivity of $GL_{n_1, n_2, \dots, n_t}(\mathbb{F}_q)$ on the set of subspaces of the same type, $N'(l_1, l_2, \dots, l_t; k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t)$ is independent of the particular choice of the subspace P of type (l_1, l_2, \dots, l_t) . By Proposition 2.2, $\mathcal{M}'(l_1, l_2, \dots, l_t; k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t)$ is non-empty if and only if (4) holds.

Corollary 2.3. *If (4) holds, then*

$$N'(l_1, l_2, \dots, l_t; k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t) = q^{\sum_{j=1}^{t-1} (k_j - k_{j+1} - l_j + l_{j+1})(n_{j+1} + \dots + n_t - k_{j+1})} \begin{bmatrix} n_t - l_t \\ k_t - l_t \end{bmatrix}_q \prod_{j=1}^{t-1} \begin{bmatrix} n_j - l_j + l_{j+1} \\ k_j - k_{j+1} - l_j + l_{j+1} \end{bmatrix}_q. \quad (6)$$

PROOF. Let

$$M = \{(P, Q) \mid P \in \mathcal{M}(l_1, l_2, \dots, l_t; n_1, n_2, \dots, n_t), Q \in \mathcal{M}(k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t), P \subseteq Q\}.$$

By computing the size of M in two ways, we have

$$\begin{aligned} & N'(l_1, l_2, \dots, l_t; k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t) N(l_1, l_2, \dots, l_t; n_1, n_2, \dots, n_t) \\ &= N(l_1, l_2, \dots, l_t; k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t) N(k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t). \end{aligned}$$

By Theorem 2.2, (6) holds. \square

3. Suborbits

Let G be a transitive permutation group on a finite set Ω , denoted by (G, Ω) . For a fixed element $a \in \Omega$, the orbits of G_a on Ω are said to be the *suborbits* of (G, Ω) , and the number of such suborbits is the *rank* of (G, Ω) . The size of each suborbit is said to be its *length*. The results on suborbits of classical groups on the set of subspaces may be found in Wang and Wei ([11]), Wei and Wang ([12, 13]), Guo, Wang and Li [2, 3, 4, 5]).

Let

$$U = \begin{pmatrix} U_{11} & & \\ & U_{22} & \\ & & U_{33} \end{pmatrix}_{\substack{k_1-k_2 \\ k_2-k_3 \\ k_3}}^{n_1 \quad n_2 \quad n_3} = \begin{pmatrix} (I \ 0) & & \\ & (I \ 0) & \\ & & (I \ 0) \end{pmatrix}_{\substack{k_1-k_2 \\ k_2-k_3 \\ k_3}}^{n_1 \quad n_2 \quad n_3} \in \mathcal{M}(k_1, k_2, k_3; n_1, n_2, n_3)$$

and let G_U be the stabilizer of U in $GL_{n_1, n_2, n_3}(\mathbb{F}_q)$. In order to determine the suborbits of $(GL_{n_1, n_2, n_3}(\mathbb{F}_q), \mathcal{M}(k_1, k_2, k_3; n_1, n_2, n_3))$, we only need to consider the orbits of G_U on $\mathcal{M}(k_1, k_2, k_3; n_1, n_2, n_3)$.

Theorem 3.1. *Let $0 \leq k_3 \leq n_3, 0 \leq k_2 - k_3 \leq n_2$ and $0 \leq k_1 - k_2 \leq n_1$. Two elements of $\mathcal{M}(k_1, k_2, k_3; n_1, n_2, n_3)$*

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ & Q_{22} & Q_{23} \\ & & Q_{33} \end{pmatrix}_{\substack{k_1-k_2 \\ k_2-k_3 \\ k_3}}^{n_1 \quad n_2 \quad n_3}, \quad P = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ & P_{22} & P_{23} \\ & & P_{33} \end{pmatrix}_{\substack{k_1-k_2 \\ k_2-k_3 \\ k_3}}^{n_1 \quad n_2 \quad n_3},$$

fall into the same orbit of G_U if and only if

$$\left. \begin{aligned} &\dim(U_{ii} \cap Q_{ii}) = \dim(U_{ii} \cap P_{ii}) \ (i = 1, 2, 3), \dim(U \cap Q) = \dim(U \cap P), \\ &\dim\left(\begin{pmatrix} U_{11} & 0 \\ 0 & U_{22} \end{pmatrix} \cap \begin{pmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{pmatrix}\right) = \dim\left(\begin{pmatrix} U_{11} & 0 \\ 0 & U_{22} \end{pmatrix} \cap \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix}\right), \\ &\dim\left(\begin{pmatrix} U_{22} & 0 \\ 0 & U_{33} \end{pmatrix} \cap \begin{pmatrix} Q_{22} & Q_{23} \\ 0 & Q_{33} \end{pmatrix}\right) = \dim\left(\begin{pmatrix} U_{22} & 0 \\ 0 & U_{33} \end{pmatrix} \cap \begin{pmatrix} P_{22} & P_{23} \\ 0 & P_{33} \end{pmatrix}\right). \end{aligned} \right\} \quad (7)$$

PROOF. Suppose Q and P are in the same orbit of G_U . Then there exists a

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ & T_{22} & T_{23} \\ & & T_{33} \end{pmatrix} \in G_U$$

such that

$$QT = \begin{pmatrix} Q_{11}T_{11} & Q_{11}T_{12} + Q_{12}T_{22} & Q_{11}T_{13} + Q_{12}T_{23} + Q_{13}T_{33} \\ & Q_{22}T_{22} & Q_{22}T_{23} + Q_{23}T_{33} \\ & & Q_{33}T_{33} \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ & P_{22} & P_{23} \\ & & P_{33} \end{pmatrix} = P.$$

Hence (7) holds.

Conversely, suppose (7) holds. Let

$$\left. \begin{aligned} &\dim(U_{11} \cap Q_{11}) = k_1 - k_2 - i_1, \dim(U_{22} \cap Q_{22}) = k_2 - k_3 - i_2, \\ &\dim(U_{33} \cap Q_{33}) = k_3 - i_3, \dim(U \cap Q) = k_1 - j_1, \\ &\dim\left(\begin{pmatrix} U_{11} & 0 \\ 0 & U_{22} \end{pmatrix} \cap \begin{pmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{pmatrix}\right) = k_1 - k_3 - j_2, \\ &\dim\left(\begin{pmatrix} U_{22} & 0 \\ 0 & U_{33} \end{pmatrix} \cap \begin{pmatrix} Q_{22} & Q_{23} \\ 0 & Q_{33} \end{pmatrix}\right) = k_2 - j_3. \end{aligned} \right\} \quad (8)$$

Then U and Q have the matrix representations

$$U = \begin{pmatrix} & n_1 & n_2 & n_3 \\ U_{111} & & & \\ U_{112} & & & \\ U_{113} & & & \\ U_{114} & & & \\ & U_{221} & & \\ & U_{222} & & \\ & U_{223} & & \\ & & U_{331} & \\ & & U_{332} & \end{pmatrix} \begin{matrix} i_1 \\ i_2 + j_1 - j_2 - j_3 \\ k_1 - k_2 + j_3 - j_1 \\ j_2 - i_1 - i_2 \\ i_2 \\ k_2 - k_3 - j_3 + i_3 \\ j_3 - i_2 - i_3 \\ i_3 \\ k_3 - i_3 \end{matrix} \quad \text{and} \quad Q = \begin{pmatrix} & n_1 & n_2 & n_3 \\ Q_{111} & Q_{121} & Q_{131} & \\ U_{112} & 0 & Q_{132} & \\ U_{113} & 0 & 0 & \\ U_{114} & Q_{124} & Q_{134} & \\ & Q_{221} & Q_{231} & \\ & U_{222} & 0 & \\ & U_{223} & Q_{233} & \\ & & Q_{331} & \\ & & U_{332} & \end{pmatrix} \begin{matrix} i_1 \\ i_2 + j_1 - j_2 - j_3 \\ k_1 - k_2 + j_3 - j_1 \\ j_2 - i_1 - i_2 \\ i_2 \\ k_2 - k_3 - j_3 + i_3 \\ j_3 - i_2 - i_3 \\ i_3 \\ k_3 - i_3 \end{matrix}, \quad (9)$$

where $\text{rank } Q_{233} = j_3 - i_2 - i_3$, $\text{rank } Q_{124} = j_2 - i_1 - i_2$ and $\text{rank } Q_{132} = i_2 + j_1 - j_2 - j_3$. It follows that $U + Q$ is a subspace of type $(k_1 + j_1, k_2 + j_1 - i_1, k_3 + j_1 - j_2)$ with a matrix representation of the form

$$\begin{pmatrix} U_{111} \\ U_{112} \\ U_{113} \\ U_{114} \\ Q_{111} & Q_{121} & Q_{131} \\ & -Q_{124} & -Q_{134} \\ & U_{221} & 0 \\ & U_{222} & 0 \\ & U_{223} & 0 \\ & Q_{221} & Q_{231} \\ & & -Q_{132} \\ & & -Q_{233} \\ & & U_{331} \\ & & U_{332} \\ & & Q_{331} \end{pmatrix}.$$

Similarly, $U + P$ is also a subspace of type $(k_1 + j_1, k_2 + j_1 - i_1, k_3 + j_1 - j_2)$ with a matrix representation just like that of $U + Q$. By Theorem 2.1, there exists a $T \in GL_{n_1, n_2, n_3}(\mathbb{F}_q)$ such that $(U + P)T = U + Q$, which implies that $UT = U$ and $PT = Q$. Hence both Q and P are in the same orbit of G_U . \square

For any Q of the form (9), let $\Lambda_{(i_1, i_2, i_3, j_3 - i_2 - i_3, j_2 - i_1 - i_2, i_2 + j_1 - j_2 - j_3)}$ be the orbit of G_U containing Q . Then

$$0 \leq i_1 \leq \min\{k_1 - k_2, n_1 + k_2 - k_1\}, \quad 0 \leq i_2 \leq \min\{k_2 - k_3, n_2 + k_3 - k_2\}, \quad 0 \leq i_3 \leq \min\{k_3, n_3 - k_3\}, \quad (10)$$

$$\max\{k_2 - k_3 - i_2, k_3 - i_3\} \leq k_2 - j_3 \leq (k_2 - k_3 - i_2) + (k_3 - i_3), \quad (11)$$

$$\max\{k_1 - k_2 - i_1, k_2 - k_3 - i_2\} \leq k_1 - k_3 - j_2 \leq (k_1 - k_2 - i_1) + (k_2 - k_3 - i_2), \quad (12)$$

$$k_2 - j_3 \leq k_1 - j_1 \leq (k_1 - k_3 - j_2) + (k_2 - j_3) - (k_2 - k_3 - i_2), \quad (13)$$

$$k_3 + j_1 - j_2 \leq n_3, \quad k_2 - k_3 + j_2 - i_1 \leq n_2. \quad (14)$$

By (11)-(14),

$$\begin{aligned} 0 &\leq j_3 - i_2 - i_3 \leq \min\{k_3 - i_3, k_2 - k_3 - i_2\}, \\ 0 &\leq j_2 - i_1 - i_2 \leq \min\{k_2 - k_3 - i_2, k_1 - k_2 - i_1, n_2 + k_3 - k_2 - i_2\}, \\ 0 &\leq i_2 + j_1 - j_2 - j_3 \leq \min\{k_1 - k_2 + i_2 - j_2, n_3 - k_3 + i_2 - j_3\}. \end{aligned}$$

Then we have

$$\left. \begin{aligned} 0 \leq i_1 &\leq \min\{k_1 - k_2, n_1 + k_2 - k_1\}, \\ 0 \leq i_2 &\leq \min\{k_2 - k_3, n_2 + k_3 - k_2\}, \\ 0 \leq i_3 &\leq \min\{k_3, n_3 - k_3\}, \\ 0 \leq j_3 - i_2 - i_3 &\leq \min\{k_3 - i_3, k_2 - k_3 - i_2\}, \\ 0 \leq j_2 - i_1 - i_2 &\leq \min\{k_2 - k_3 - i_2, k_1 - k_2 - i_1, n_2 + k_3 - k_2 - i_2\}, \\ 0 \leq i_2 + j_1 - j_2 - j_3 &\leq \min\{k_1 - k_2 + i_2 - j_2, n_3 - k_3 + i_2 - j_3\}. \end{aligned} \right\} \quad (15)$$

Conversely, for any given integers i_1, i_2, i_3, j_1, j_2 and j_3 satisfying (15), pick

$$Q = \begin{pmatrix} i_1 & k_1 - k_2 - i_1 & i_1 & n_1 - k_1 + k_2 - i_1 & i_2 & k_2 - k_3 - i_2 & i_2 & n_2 - k_2 + k_3 - i_2 & i_3 & k_3 - i_3 & i_3 & n_3 - k_3 - i_3 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & C \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & B \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \end{pmatrix} \begin{matrix} k_1 - k_2 - i_1 \\ i_1 \\ k_2 - k_3 - i_2 \\ i_2 \\ k_3 - i_3 \\ i_3 \end{matrix},$$

where

$$A = \begin{pmatrix} 0 & 0 \\ I^{(j_2 - i_1 - i_2)} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ I^{(j_3 - i_2 - i_3)} & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & I^{(i_2 + j_1 - j_2 - j_3)} \\ 0 & 0 \end{pmatrix}.$$

Then $Q \in \Lambda_{(i_1, i_2, i_3, j_3 - i_2 - i_3, j_2 - i_1 - i_2, i_2 + j_1 - j_2 - j_3)}$; and so the orbit $\Lambda_{(i_1, i_2, i_3, j_3 - i_2 - i_3, j_2 - i_1 - i_2, i_2 + j_1 - j_2 - j_3)}$ exists. It follows that the orbits of G_U are completely determined by $(i_1, i_2, i_3, j_3 - i_2 - i_3, j_2 - i_1 - i_2, i_2 + j_1 - j_2 - j_3)$ satisfying (15). Therefore, we have the following result.

Theorem 3.2. *Let $0 \leq k_1 - k_2 \leq n_1, 0 \leq k_2 - k_3 \leq n_2$ and $0 \leq k_3 \leq n_3$. Then the number of suborbits of $(GL_{n_1, n_2, n_3}(\mathbb{F}_q), \mathcal{M}(k_1, k_2, k_3; n_1, n_2, n_3))$ is*

$$\sum_{i_1=0}^{\min\{k_1 - k_2, n_1 + k_2 - k_1\}} \sum_{i_2=0}^{\min\{k_2 - k_3, n_2 + k_3 - k_2\}} \sum_{i_3=0}^{\min\{k_3, n_3 - k_3\}} \sum_{j_3=i_2+i_3}^{\min\{k_3+i_2, k_2-k_3+i_3\}} \sum_{j_2=i_1+i_2}^{\min\{k_2-k_3+i_1, k_1-k_2+i_2, n_2+k_3-k_2+i_1\}} (1 + \min\{k_1 - k_2 + i_2 - j_2, n_3 - k_3 + i_2 - j_3\}).$$

In order to compute the length of suborbits of $(GL_{n_1, n_2, n_3}(\mathbb{F}_q), \mathcal{M}(k_1, k_2, k_3; n_1, n_2, n_3))$, we need the following results.

Proposition 3.3. ([7, Chapter 1, Theorem 5]) *The number of $m \times n$ matrices with rank i over \mathbb{F}_q is*

$$N(i; m \times n) = q^{i(i-1)/2} \begin{bmatrix} m \\ i \end{bmatrix}_q \prod_{t=n-i+1}^n (q^t - 1).$$

Proposition 3.4. ([7, Chapter 6, Theorem 7]) *Let $1 \leq m \leq n$ and $0 \leq i \leq \min\{m, n - m\}$. For a given m -dimensional subspace P of \mathbb{F}_q^n , the number of m -dimensional subspaces intersecting P at $(m - i)$ -dimensional subspaces of \mathbb{F}_q^n is*

$$q^{i^2} \begin{bmatrix} n - m \\ i \end{bmatrix}_q \begin{bmatrix} m \\ i \end{bmatrix}_q.$$

Proposition 3.5. ([8, Lemma 2.4]) *For any $m_1 \times n$ matrix A_1 with rank t_1 , the number of $m_2 \times n$ matrix A_2 satisfying $\text{rank} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = t_2$ is $q^{m_2 t_1} N(t_2 - t_1; m_2 \times (n - t_1))$.*

Lemma 3.6. *For any $m \times n_1$ matrix A_1 with rank t_1 , the number of $m \times n_2$ matrix A_2 satisfying $\text{rank}(A_1 \ A_2) = t_2$ is $q^{t_1 n_2} N(t_2 - t_1; (m - t_1) \times n_2)$.*

PROOF. The proof is similar to that of [8, Lemma 2.4], and will be omitted. \square

Lemma 3.7. *The number of $(m_1 + m_2) \times (n_1 + n_2)$ matrix*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}_{m_1 \atop m_2}^{n_1 \atop n_2}$$

with $\text{rank}(C \ D) = \alpha$ and $\text{rank} \begin{pmatrix} B \\ D \end{pmatrix} = \alpha$ is

$$\sum_{l=\max\{0, \alpha-n_1, \alpha-m_1\}}^{\alpha} q^{(m_1+n_1)l+m_1n_1} N(l; m_2 \times n_2) N(\alpha-l; m_1 \times (n_2-l)) N(\alpha-l; (m_2-l) \times n_1).$$

PROOF. Let $\text{rank } D = l$. Then $\max\{0, \alpha - n_1, \alpha - m_1\} \leq l \leq \alpha$. By Proposition 3.3, there are $N(l; m_2 \times n_2)$ choices for D . For a given D , by Proposition 3.5 there are $q^{m_1 l} N(\alpha - l; m_1 \times (n_2 - l))$ choices for B , by Lemma 3.6 there are $q^{l n_1} N(\alpha - l; (m_2 - l) \times n_1)$ choices for C . Note that there are $q^{m_1 n_1}$ choices for A . Therefore, the desired result follows. \square

Theorem 3.8. *Suppose (15) holds. Then the length of the suborbit $\Lambda_{(i_1, i_2, i_3, j_3-i_2-i_3, j_2-i_1-i_2, i_2+j_1-j_2-j_3)}$ of $(GL_{n_1, n_2, n_3}(\mathbb{F}_q))$, $\mathcal{M}(k_1, k_2, k_3; n_1, n_2, n_3)$ is*

$$\begin{aligned} & q^{(n_2+n_3-k_2)i_1+(n_3+k_1-k_2-k_3-i_1)i_2+(k_1-k_3-i_1-i_2)i_3+i_1^2+i_2^2+i_3^2} \begin{bmatrix} n_1+k_2-k_1 \\ i_1 \end{bmatrix}_q \begin{bmatrix} k_1-k_2 \\ i_1 \end{bmatrix}_q \begin{bmatrix} n_2+k_3-k_2 \\ i_2 \end{bmatrix}_q \begin{bmatrix} k_2-k_3 \\ i_2 \end{bmatrix}_q \begin{bmatrix} n_3-k_3 \\ i_3 \end{bmatrix}_q \begin{bmatrix} k_3 \\ i_3 \end{bmatrix}_q \\ & \times N(j_3 - i_2 - i_3; (k_2 - k_3 - i_2) \times (n_3 - k_3 - i_3)) N(j_2 - i_1 - i_2; (k_1 - k_2 - i_1) \times (n_2 + k_3 - k_2 - i_2)) \\ & \times \sum_{l=\max\{0, 2i_2+j_1-j_2-2j_3+i_3, 2i_2+j_1-2j_2-j_3+i_1\}}^{i_2+j_1-j_2-j_3} q^{(j_2+j_3-i_1-i_3)l+(j_2-i_1-i_2)(j_3-i_2-i_3)} N(l; (k_1 - k_2 - j_2 + i_2) \times (n_3 - k_3 - j_3 + i_2)) \\ & \times N(i_2 + j_1 - j_2 - j_3 - l; (j_2 - i_1 - i_2) \times (n_3 - k_3 - j_3 + i_2 - l)) N(i_2 + j_1 - j_2 - j_3 - l; (k_1 - k_2 - j_2 + i_2 - l) \times (j_3 - i_2 - i_3)). \end{aligned}$$

PROOF. Suppose

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ 0 & P_{22} & P_{23} \\ 0 & 0 & P_{33} \end{pmatrix}_{k_1-k_2 \atop k_2-k_3 \atop k_3}^{n_1 \atop n_2 \atop n_3} \in \Lambda_{(i_1, i_2, i_3, j_3-i_2-i_3, j_2-i_1-i_2, i_2+j_1-j_2-j_3)}.$$

Then P_{11} is a $(k_1 - k_2)$ -dimensional subspace of $\mathbb{F}_q^{n_1}$ such that $\dim(P_{11} \cap U_{11}) = k_1 - k_2 - i_1$, P_{22} is a $(k_2 - k_3)$ -dimensional subspace of $\mathbb{F}_q^{n_2}$ such that $\dim(P_{22} \cap U_{22}) = k_2 - k_3 - i_2$ and P_{33} is a k_3 -dimensional subspace of $\mathbb{F}_q^{n_3}$ such that $\dim(P_{33} \cap U_{33}) = k_3 - i_3$. By Proposition 3.4, there are

$$\alpha = q^{i_1^2+i_2^2+i_3^2} \begin{bmatrix} n_1+k_2-k_1 \\ i_1 \end{bmatrix}_q \begin{bmatrix} k_1-k_2 \\ i_1 \end{bmatrix}_q \begin{bmatrix} n_2+k_3-k_2 \\ i_2 \end{bmatrix}_q \begin{bmatrix} k_2-k_3 \\ i_2 \end{bmatrix}_q \begin{bmatrix} n_3-k_3 \\ i_3 \end{bmatrix}_q \begin{bmatrix} k_3 \\ i_3 \end{bmatrix}_q$$

choices for (P_{11}, P_{22}, P_{33}) . By the transitivity of G_U on $\Lambda_{(i_1, i_2, i_3, j_3-i_2-i_3, j_2-i_1-i_2, i_2+j_1-j_2-j_3)}$, we may pick

$$P_{11} = (0^{(k_1-k_2, i_1)} I^{(k_1-k_2)} 0^{(k_2-k_1, n_1+k_2-k_1-i_1)}), P_{22} = (0^{(k_2-k_3, i_2)} I^{(k_2-k_3)} 0^{(k_2-k_3, n_2+k_3-k_2-i_2)}) \text{ and } P_{33} = (0^{(k_3, i_3)} I^{(k_3)} 0^{(k_3, n_3-k_3-i_3)}).$$

Then P_{12}, P_{23}, P_{13} have the matrix representations of the forms

$$P_{12} = \begin{pmatrix} i_2 & k_2-k_3 & n_2+k_3-k_2-i_2 \\ A_{11} & 0 & A_{12} \\ A_{21} & 0 & A_{22} \end{pmatrix}_{k_1-k_2-i_1 \atop i_1}^{i_2 \atop k_2-k_3 \atop n_2+k_3-k_2-i_2}, P_{23} = \begin{pmatrix} i_3 & k_3 & n_3-k_3-i_3 \\ B_{11} & 0 & B_{12} \\ B_{21} & 0 & B_{22} \end{pmatrix}_{k_2-k_3-i_2 \atop i_2}^{i_3 \atop k_3 \atop n_3-k_3-i_3}, P_{13} = \begin{pmatrix} i_3 & k_3 & n_3-k_3-i_3 \\ C_{11} & 0 & C_{12} \\ C_{21} & 0 & C_{22} \end{pmatrix}_{k_1-k_2-i_1 \atop i_1}^{i_3 \atop k_3 \atop n_3-k_3-i_3},$$

where $\text{rank } A_{12} = j_2 - i_1 - i_2$, $\text{rank } B_{12} = j_3 - i_2 - i_3$, $\text{rank}(A_{12} \ C_{12}) = j_1 - j_3 - i_1$ and $\text{rank} \begin{pmatrix} C_{12} \\ B_{12} \end{pmatrix} = j_1 - j_2 - i_3$. By Proposition 3.3, there are $N(j_3 - i_2 - i_3; (k_2 - k_3 - i_2) \times (n_3 - k_3 - i_3))$ choices for B_{12} , and there are $N(j_2 - i_1 - i_2; (k_1 - k_2 - i_1) \times (n_2 + k_3 - k_2 - i_2))$ choices for A_{12} .

Now we compute the numbers of C_{12} satisfying the above conditions. Let M be the set of all matrices of the forms

$$\begin{pmatrix} & n_2+k_3-k_2-i_2 & n_3-k_3-i_3 \\ A_{12} & C_{12} \\ 0 & B_{12} \end{pmatrix}_{\substack{k_1-k_2-i_1 \\ k_2-k_3-i_2}}, \quad (16)$$

where $\text{rank } A_{12} = j_2 - i_1 - i_2$, $\text{rank } B_{12} = j_3 - i_2 - i_3$, $\text{rank } (A_{12} \ C_{12}) = j_1 - j_3 - i_1$ and $\text{rank } \begin{pmatrix} C_{12} \\ B_{12} \end{pmatrix} = j_1 - j_2 - i_3$. Let \mathcal{G} (resp. \mathcal{S}) be the set of all non-singular matrices of the forms

$$\begin{pmatrix} & k_1-k_2-i_1 & k_2-k_3-i_2 \\ T_{11} & \\ & T_{22} \end{pmatrix}_{\substack{k_1-k_2-i_1 \\ k_2-k_3-i_2}} \quad \text{resp.} \quad \begin{pmatrix} & n_2+k_3-k_2-i_2 & n_3-k_3-i_3 \\ S_{11} & \\ & S_{22} \end{pmatrix}_{\substack{n_2+k_3-k_2-i_2 \\ n_3-k_3-i_3}}.$$

There is an action of $\mathcal{G} \times \mathcal{S}$ on M defined as follows:

$$\begin{aligned} M \times (\mathcal{G} \times \mathcal{S}) &\longrightarrow M \\ (A, (T, S)) &\longmapsto T^{-1}AS. \end{aligned}$$

For a given (A_{12}, B_{12}) , there exist

$$\begin{pmatrix} & k_1-k_2-i_1 & k_2-k_3-i_2 \\ T_{11} & \\ & T_{22} \end{pmatrix}_{\substack{k_1-k_2-i_1 \\ k_2-k_3-i_2}} \in \mathcal{G} \text{ and } \begin{pmatrix} & n_2+k_3-k_2-i_2 & n_3-k_3-i_3 \\ S_{11} & \\ & S_{22} \end{pmatrix}_{\substack{n_2+k_3-k_2-i_2 \\ n_3-k_3-i_3}} \in \mathcal{S}$$

such that

$$\begin{pmatrix} T_{11} & \\ & T_{22} \end{pmatrix} \begin{pmatrix} A_{12} & C_{12} \\ & B_{12} \end{pmatrix} \begin{pmatrix} S_{11} & \\ & S_{22} \end{pmatrix} = \begin{pmatrix} & j_2-i_1-i_2 & n_2+k_3-k_2-j_2+i_1 & j_3-i_2-i_3 & n_3-k_3-j_3+i_2 \\ I & 0 & C_1 & C_2 \\ & 0 & C_3 & C_4 \\ & & I & 0 \\ & & & 0 \end{pmatrix}_{\substack{j_2-i_1-i_2 \\ k_1-k_2-j_2+i_2 \\ j_3-i_2-i_3 \\ k_2-k_3-j_3+i_2}}, \quad (17)$$

where

$$\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} = T_{11}C_{12}S_{22}, \quad \text{rank } (C_3 \ C_4) = \text{rank } \begin{pmatrix} C_2 \\ C_4 \end{pmatrix} = i_2 + j_1 - j_2 - j_3.$$

Therefore, for a given (A_{12}, B_{12}) , the number of C_{12} satisfying (16) is equal to the number of $\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$ satisfying

(17). By Lemma 3.7, the number of $\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$ satisfying (17) is

$$\sum_{l=\max\{0, 2i_2+j_1-j_2-2j_3+i_3, 2i_2+j_1-2j_2-j_3+i_1\}}^{i_2+j_1-j_2-j_3} q^{(j_2+j_3-i_1-i_3)l+(j_2-i_1-i_2)(j_3-i_2-i_3)} N(l; (k_1-k_2-j_2+i_2) \times (n_3-k_3-j_3+i_2)) \\ \times N(i_2+j_1-j_2-j_3-l; (j_2-i_1-i_2) \times (n_3-k_3-j_3+i_2-l)) N(i_2+j_1-j_2-j_3-l; (k_1-k_2-j_2+i_2-l) \times (j_3-i_2-i_3)).$$

Note that there are $q^{(n_2+n_3-k_2)i_1+(n_3+k_1-k_2-k_3-i_1)i_2+(k_1-k_3-i_1-i_2)i_3}$ choices for $(A_{11}, A_{21}, A_{22}, B_{11}, B_{21}, B_{22}, C_{11}, C_{21}, C_{22})$. Hence the desired result follows. \square

Remarks.

(i) Dam and Koolen [1] constructed the twisted Grassmann graph $\tilde{J}_q(2e+1, e)$, which is the first know family of non-vertex-transitive distance-regular graphs with unbounded diameter. Pick the hyperplane $H = (0^{(1,2e)} \ I^{(2e)})$ in \mathbb{F}_q^{2e+1} . Then $P_0 = (0^{(e-1, e+2)} \ I^{(e-1)})$ is a vertex of the twisted Grassmann graph. Note that the last subconstituent of $\tilde{J}_q(2e+1, e)$ about P_0 is just $\mathcal{M}(e+1, e, 0; 1, e+1, e-1)$.

(ii) Similarly, we may determine all the suborbits of $(GL_{n_1, n_2, \dots, n_t}(\mathbb{F}_q), \mathcal{M}(k_1, k_2, \dots, k_t; n_1, n_2, \dots, n_t))$ for any $t \geq 4$.

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